

A Massive Non-Abelian Vector Model

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Abstract

The introduction of a Lagrange multiplier field to ensure that the classical equations of motion are satisfied serves to restrict radiative corrections in a model to being only one loop. The consequences of this for a massive non-Abelian vector model are considered.

Keywords: Massive Vector

The elusiveness of the Higgs Boson has led to reconsideration of various ways of endowing a non-Abelian vector field with a mass. For a $U(1)$ vector field, Stueckelberg has shown that such a mass can be inserted “by hand” without compromising either unitarity or renormalizability [1]. Indeed, the $U(1)$ sector of the Standard Model may have such a mass, which makes the masslessness of the photon somewhat mysterious [2,3].

The Lagrangian

$$\mathcal{L}_I(A) = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - \frac{m^2}{2}A_\mu^a A^{a\mu} \quad (1)$$

$$([D_\mu, D_\nu]^{ab} = c^{apb}F_{\mu\nu}^p, D_\mu^{ab} = \partial_\mu\delta^{ab} + c^{apb}A_\mu^p, \eta^{\mu\nu} = \text{diag}(-, +, +, +))$$

has been investigated with the hope that the symmetry present would be sufficient to ensure that this model for the vector A_μ^a is both unitary and renormalizable for $m^2 \neq 0$ even if the group were not $U(1)$ [4-23]. It has been shown that tree level unitarity is not upheld on account of the longitudinal polarization of A_μ^a [20-22] and that renormalizability is lost beyond one loop order [7, 12].

The equation of motion for A_μ^a is

$$D_\mu^{ab}(A)F^{b\mu\nu} - m^2 A^{a\nu} = 0; \quad (2)$$

we can ensure that $A^{a\mu}$ satisfies this equation of motion by supplementing \mathcal{L}_I with

$$\mathcal{L}_{II}(A, B) B_\nu^a (D_\mu^{ab} F^{b\mu\nu} - m^2 A^{a\nu}) \quad (3)$$

where B_ν^a is a ‘‘Lagrange multiplier’’ field. The Lagrangian $\mathcal{L} = \mathcal{L}_I + \mathcal{L}_{II}$ has been first investigated when $m^2 = 0$ in [24] and also later in [25]. In [24], it has been shown that perturbative radiative effects vanish beyond one loop order and that consequently the model can be considered to be ‘‘solvable’’. We wish to now extend these considerations to the case $m^2 \neq 0$.

When $m^2 \neq 0$, no local gauge symmetry is present and so the generating functional is simply

$$Z[J_\mu^a, K_\mu^a] = \int DA_\mu^a DB_\mu^a \exp i \int d^4x [\mathcal{L} + J_\mu^a A^{a\mu} + K_\mu^a B^{a\mu}]. \quad (4)$$

The terms in the action that are bilinear in the fields are

$$\frac{1}{2} (A_\mu, B_\mu^a) \begin{pmatrix} a^{\mu\nu} & a^{\mu\nu} \\ a^{\mu\nu} & 0 \end{pmatrix} \begin{pmatrix} A_\nu^a \\ B_\nu^a \end{pmatrix} \quad (5)$$

where $a^{\mu\nu} = (\partial^2 - m^2)\eta^{\mu\nu} - \partial^\mu \partial^\nu$. The inverse of the operator \mathbf{M} in eq. (5) is

$$\mathbf{M}^{-1} = \begin{pmatrix} 0 & (a^{-1})_{\mu\nu} \\ (a^{-1})_{\mu\nu} & -(a^{-1})_{\mu\nu} \end{pmatrix} \quad (6)$$

where

$$(a^{-1})_{\mu\nu} = \frac{\eta_{\mu\nu} - \partial_\mu \partial_\nu / m^2}{\partial^2 - m^2}. \quad (7)$$

From eq. (6), we see that there is a propagator $\langle BB \rangle$ for the field B_μ^a as well as mixed propagators $\langle AB \rangle$ and $\langle BA \rangle$, but no $\langle AA \rangle$ propagator for the field A_μ^a . This fact, combined having only the vertices $\langle AAA \rangle$, $\langle AAAA \rangle$, $\langle BAA \rangle$, and $\langle BAAA \rangle$ (ie, no vertex involves more than one external B_μ^a field), leads to the disappearance of all loop diagrams beyond one loop order, as in the $m^2 = 0$ case [24]. The one loop diagrams receive no contribution from the propagator $\langle BB \rangle$ or from the vertices $\langle AAA \rangle$, $\langle AAAA \rangle$.

To see more directly how diagrams beyond one loop order cannot contribute to Z , we first consider the path integral over B_μ^a in eq. (4) to give

$$Z[J_\mu^a, K_\mu^a] = \int DA_\mu^a \delta \left(\frac{\delta \mathcal{L}_I(A)}{\delta A_\lambda^p} + K^{a\lambda} \right) \exp i \int d^4x (\mathcal{L}_I(A) + J_\mu^a A^{a\mu}). \quad (8)$$

The standard result

$$\int_{-\infty}^{\infty} dx \delta(f(x)) g(x) = \sum_i g(a_i) / f'(a_i) \quad (f(a_i) = 0) \quad (9)$$

can be used to evaluate the path integral over A_μ^a in eq. (8). We obtain

$$Z[J_\mu^a, K_\mu^a] = \sum_i \left(\det \frac{\delta^2 \mathcal{L}_I(A)}{\delta A_\lambda^p \delta A_\sigma^q} \right)^{-1} \exp i \int d^4x (\mathcal{L}_I(A) + J_\mu^a A^{a\mu}) \quad (10)$$

where the sum in eq. (10) is over those configurations that satisfy the equation of motion

$$\frac{\delta \mathcal{L}_I}{\delta A_\lambda^p} + K^{p\lambda} = D_\rho^{pq} F^{q\rho\lambda} - m^2 A^{p\lambda} + K^{p\lambda} = 0. \quad (11)$$

We can relate the result of eq. (10) with what is obtained by quantizing the action involving only the Lagrangian \mathcal{L}_I of eq. (1). If we consider

$$\bar{Z}[K_\mu^a] = \int DA_\mu^a \exp i \int d^4x [\mathcal{L}_I(A) + K_\mu^a A^{a\mu}] \quad (12)$$

and expand [28-29]

$$A_\mu^a = V_\mu^a + Q_\mu^a \quad (13)$$

where V_μ^a is a solution to the classical equation

$$\frac{\delta \mathcal{L}_I}{\delta A_\mu^a} + K_\mu^a = 0 \quad (14)$$

and Q_μ^a is a fluctuation about V_μ^a , then working to the term quadratic in Q_μ^a in the action we obtain

$$\bar{Z}[J_\mu^a, V_\mu^a] \approx \int DQ_\mu^a \exp i \int d^4x \left[\mathcal{L}_I(V) + \frac{1}{2} Q_\mu^a \frac{\delta^2 \mathcal{L}_I(V)}{\delta Q_\mu^a \delta Q_\nu^b} Q_\nu^b + K_\mu^a V^{a\mu} \right]. \quad (15)$$

The functional integral over Q_μ^a can be evaluated in eq. (15) to yield

$$\approx \left(\det \frac{\delta^2 \mathcal{L}_I(V)}{\delta Q_\mu^a \delta Q_\nu^b} \right)^{-1/2} \exp i \int d^4x [\mathcal{L}_I(V) + K_\mu^a V^{a\mu}]. \quad (16)$$

It is known that the exponential in eq. (16) is a consequence of tree level diagrams while the functional determinant is a consequence of the one loop diagrams in the presence of a background field V_μ^a [26, 27].

Eqs. (10) and (16) differ in two respects. First of all, one must set $J_\mu^a = K_\mu^a$ in eq. (10). Secondly, as the connected Green's functions are generated by $W = -i \ln Z$, it follows that the connected one loop Green's functions that follow from eq. (10) differ from those that follow from eq. (16) by a factor of $1/2$.

The arguments used to examine the unitarity of tree amplitudes that follow from eq. (12) [20] rely on examining their high energy behaviour when the wave function associated with the external legs are plane wave states. However, these plane wave states are just the basis states for the decomposition of the classical fields that contribute to the argument of the exponentials appearing in eqs. (10) and (16). These exponentials incorporate the sum of all the tree diagrams in which there is a source function; we may consequently surmise that once one sums all the tree level diagrams, one does not suffer from the problems with unitarity when $m^2 \neq 0$ discussed in ref. [20-22].

The renormalizability of the model described solely by the Lagrangian $\mathcal{L}_I(A)$ has been examined in refs. [4-23]. By comparing eqs. (10) and (16), we see that the connected Green's functions at one loop order (whose generating functional is given by $W = -i \ln Z$ [28, 29]) for the models defined by \mathcal{L}_I and $\mathcal{L}_I + \mathcal{L}_{II}$ differ by a factor of 1/2. Consequently, if the model defined by \mathcal{L}_I is renormalizable at one loop order, then the model defined by $\mathcal{L}_I + \mathcal{L}_{II}$ is completely renormalizable.

Any discussion of renormalizability based on a direct analysis of Feynman diagrams which follow immediately from \mathcal{L}_I is complicated by the term $(k_\mu k_\nu / m^2) / (k^2 + m^2)$ appearing in the vector propagator (of eq. (7)). This is because the usual power counting arguments normally used to establish renormalizability breakdown. Even if operator regularization [17] were used to analyze the model, the exponential arising from the bilinear term in the vector field that appears in \mathcal{L}_I

$$\exp -i[k^2 L_{\mu\nu} + m^2 \eta_{\mu\nu}]t = e^{-im^2 t} \left[L_{\mu\nu} e^{-ik^2 t} + T_{\mu\nu} \right] \quad (17)$$

($T_{\mu\nu} \equiv k_\mu k_\nu / k^2 \equiv \eta_{\mu\nu} - L_{\mu\nu}$) the term involving $T_{\mu\nu}$ does not contain the necessary damping factor of $e^{-ik^2 t}$, making use of this regulating technique problematic.

This difficulty can be overcome [7, 12] by adapting the Faddeev-Popov approach to quantizing the massless Yang-Mills theory [30]. By inserting in turn the constant factors

$$\text{const} = \int D\Omega \delta(\partial_\mu A^{\Omega\mu} - k) \Delta_{FP} \quad (18a)$$

$$\text{const} = \int dk e^{\frac{-i}{2\alpha} \int dx k^2} \quad (18b)$$

into the generating functional of eq. (12) ($A_\mu = T^a A_\mu^a$, $A_\mu^\Omega = \Omega(A_\mu + \partial_\mu) \Omega^{-1}$, $\Omega = e^{i\phi}$, $\Delta_{FP} = \det(\partial^\mu (\partial_\mu \delta^{ab} + c^{ab} A_\mu^b))$) and performing the transformation $A_\mu^\Omega \rightarrow A_\mu$ the generating functional of eq. (12) becomes

$$\begin{aligned} \bar{Z}[K_\mu^a] = \int D\Omega \int DA_\mu^a \Delta_{FP} \exp i \int dx \left[-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\alpha} (\partial \cdot A^a)^2 \right. \\ \left. - \frac{m^2}{2} \text{Tr}(A_\mu + \Omega \partial_\mu \Omega^{-1})^2 + K_\mu^a A^{a\mu} \right]. \end{aligned} \quad (19)$$

If only the lowest order contribution in ϕ is retained in eq. (19), then the integral over ϕ can be evaluated and the one loop generating functional is found to be

$$\begin{aligned} \bar{Z}[K_\mu^a] \approx \int DA_\mu^a \Delta_{FP}^{1/2} \exp i \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\alpha} (\partial \cdot A^a)^2 \right. \\ \left. - \frac{m^2}{2} A_\mu^a A^{a\mu} + K_\mu^a A^{a\mu} \right]. \end{aligned} \quad (20)$$

Any (one loop) divergences arising from this functional integral can be renormalized [7, 12]. (Indeed, it is shown in ref. [30] that eq. (20) also arises in the Yang-Mills-Higgs model at one loop order

if the Higgs field were to be integrated out.) As a result, we conclude that the model defined by $\mathcal{L}_I + \mathcal{L}_{II}$ is renormalizable.

In refs. [15-17] the Kunimasa-Goto [4] form of the measure Yang-Mills model is examined and it is purportedly shown that even at one loop order the model is not renormalizable. However, this analysis is likely deficient as the implementation of background field quantization in these papers for the “Stueckelberg” field is not consistent with the analysis of this technique provided in refs. [28, 29]; consequently it is not clear if in fact it is one loop effects that are being considered in refs. [15-17].

A canonical analysis of the system described by \mathcal{L} is straightforward. The momenta that follow from

$$\pi^{a\mu} = \partial\mathcal{L}/\partial(\partial_0 A_\mu^a) \quad \tau^{a\mu} = \partial\mathcal{L}/\partial(\partial_0 B_\mu^a)$$

are

$$\pi^{a0} = \tau^{a0} = 0 \tag{21a, b}$$

$$\pi^{ai} = F_{0i}^a - D_i^{ab}b^b + \partial_0 B_i^a + c^{abc}a^b B_i^c \tag{21c}$$

$$\tau^{ai} = F_{0i}^a = \partial_0 A_i^a - D_i^{ab}a^b \tag{21d}$$

$$(a^a \equiv A_0^a, \quad b^a \equiv B_0^a)$$

which lead to the canonical Hamiltonian

$$\begin{aligned} \mathcal{H} = & -\frac{1}{2}(\pi^{ai} - \tau^{ai})^2 + \frac{1}{2}\pi^{ai}\pi^{ai} + \frac{1}{4}F_{ij}^a F_{ij}^a + (D_i^{ab}B_j^b)(F_{ij}^a) \\ & -a^a(D_i^{ab}\pi^{bi} + c^{abc}B_i^b\tau^{ai}) - b^a D_i^{ab}\tau^{bi} \\ & + \frac{m^2}{2}[(A^{ai} + B^{ai})^2 - B^{ai}B^{ai} - (a^a + b^a)^2 + b^a b^a]. \end{aligned} \tag{22}$$

The primary constraints of eqs. (21a,b) lead to the secondary constraints

$$D_i^{ab}\pi^{bi} + c^{abc}\tau^{ai} + m^2(a^a + b^a) = 0 \tag{23a}$$

$$D_i^{ab}\tau^{bi} + m^2 a^a = 0; \tag{23b}$$

all constraints are second class provided $m^2 \neq 0$ [32]. This Hamiltonian is not positive definite.

In order to incorporate a Fermion ψ , we supplement \mathcal{L}_I of eq. (1) with

$$\mathcal{L}'_I = \bar{\psi}(\not{p} - \not{A}^a T^a - \kappa)\psi \tag{24a}$$

and \mathcal{L}_{II} of eq. (3) with

$$\mathcal{L}'_{II} = -\bar{\psi}\not{B}^a T^a \psi + \bar{\eta}(\not{p} - \not{A}^a T^a - \kappa)\psi + \bar{\psi}(\not{p} - \not{A}^a T^a - \kappa)\eta \tag{24b}$$

where $\bar{\eta}$ and η are, like B_μ^a , Lagrange multiplier fields used to ensure that the equations of motion that follow from \mathcal{L}'_I are satisfied. Again, no effects beyond one loop order arise when considering $\mathcal{L}_I + \mathcal{L}'_I + \mathcal{L}_{II} + \mathcal{L}'_{II}$.

To illustrate this, let us consider the Abelian limit of eqs. (1,3,24) so that

$$\begin{aligned} \mathcal{L} = & \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{m^2}{2} A_\mu^2 + B_\nu [\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) - m^2 A^\nu] \\ & + \bar{\psi}(\not{p} - \not{A} - \kappa)\psi - \bar{\psi} \bar{B} \psi + \bar{\eta}(\not{p} - \not{A} - \kappa)\psi + \bar{\psi}(\not{p} - \not{A} - \kappa)\eta. \end{aligned} \quad (25)$$

The bilinear terms in eq. (15) can be written as

$$\mathcal{L}_2 = \frac{1}{2} (A_\mu, B_\mu) \begin{pmatrix} a^{\mu\nu} & a^{\mu\nu} \\ a^{\mu\nu} & 0 \end{pmatrix} \begin{pmatrix} A_\nu \\ B_\nu \end{pmatrix} + (\bar{\psi}, \bar{\eta}) \begin{pmatrix} K & K \\ K & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \eta \end{pmatrix} \quad (26)$$

where $a^{\mu\nu} = (\partial^2 - m^2) \eta^{\mu\nu} - \partial^\mu \partial^\nu$ and $K = \not{p} - \kappa$. Since

$$\begin{pmatrix} a^{\mu\nu} & a^{\mu\nu} \\ a^{\mu\nu} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & a_{\mu\nu}^{-1} \\ a_{\mu\nu}^{-1} & -a_{\mu\nu}^{-1} \end{pmatrix} \quad \left(a_{\mu\nu}^{-1} = \frac{\eta_{\mu\nu} - \partial_\mu \partial_\nu / m^2}{\partial^2 - m^2} \right) \quad (27)$$

$$\begin{pmatrix} K & K \\ K & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & K^{-1} \\ K^{-1} & -K^{-1} \end{pmatrix} \quad (K^{-1} = 1/(\not{p} - \kappa)) \quad (27)$$

we see that the propagators $\langle AA \rangle$ and $\langle \psi \bar{\psi} \rangle$ do not follow from \mathcal{L} . As all vertices are at most linear in B_ν , η and $\bar{\eta}$, we see that there are no diagrams beyond one-loop order in the loop expansion for the effective action.

This approach to generating a unitary and renormalizable theory for massive vectors may be applicable to the Standard Model. The study of this case is presently in progress [33]. It may be necessary to consider such alternatives, both since the Higgs particle is proving to be difficult to detect, and since the radiative corrections to the Higgs potential appear to “flatten” it [34-36].

It has been noted [37] that in the first order form of the Einstein-Cartan action in $2+1$ dimensions, the loop expansion about a vanishing dreibein and spin connection terminates at one-loop order because the dreibein field enters the classical action only linearly. The same situation occurs when considering the first-order Einstein-Hilbert action in $1+1$ dimensions [38].

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